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Baroclinic instability of three-layer flows
Part 1. Linear stability

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The stability of quasi-geostrophic three-layer stratified flow in a channel is examined. The mean zonal velocity $\bar{U}_i$ is uniform within each layer ($i = 1, 2, 3$). Thus, as in the two-layer model of Phillips (1954), the only source of energy for growing disturbances is the potential energy stored in the sloping interfaces. Attention is focused upon the case in which $\epsilon = \Delta\rho_2/\Delta\rho_1 \ll 1$ ($\Delta\rho_1, \Delta\rho_2$ are the changes in density across the upper and the lower interfaces). Two scales of instability are possible: long waves (wavenumber $O(1)$) associated with the upper interface and short waves (wavenumber $O(\epsilon^{-\frac{1}{2}})$) associated with the lower interface. It is found that short waves are unstable only when S (the ratio of the slope of the lower interface to that of the upper interface) is greater than one or less than zero, i.e. when the gradients of potential vorticity in the two lower layers have opposite signs. The short waves have the largest growth rates when $S^2\epsilon$ (the ratio of the potential energy stored in the lower interface to that stored in the upper interface) $\gtrsim 1$. The results of this analysis are used in an accompanying paper to interpret some experiments with three-layer eddies.

1. Introduction

Baroclinic instability, a process by which the potential energy of stratified, rotating flow is transferred to growing disturbances, has been the subject of much research since Eady (1949) demonstrated that the scales of motion associated with baroclinic waves are very similar to those observed in the weather of our atmosphere. Much progress in understanding the different aspects of the instability process has been made by the study, both experimentally and analytically, of layer models. Whilst retaining many features of interest, such models, with their simple vertical structure, are more amenable to theoretical investigations and are more easily studied in the laboratory than are continuously stratified flows.

Most attention has been focused upon two-layer flows, from which it has been possible to gain much insight into baroclinic instability in more complex flows, for example, examining the effects of Ekman dissipation or small horizontal shear (see e.g. Pedlosky 1979). There are, however, some questions that cannot be answered by consideration of two-layer flows. One particular problem that will be addressed in this and an accompanying paper (Smeed 1988) is: how is the instability process modified by non-uniform vertical stratification?

Some aspects of this question may be investigated by using three-layer models such as that sketched in figure 1, in which the flow is composed of layers of depth H_i ($i = 1, 2, 3$) each of uniform density ρ_i ($\rho_1 < \rho_2 < \rho_3$) and mean zonal velocity $\bar{U}_i$.

Phillips (1954) investigated the linear stability of a similar quasi-geostrophic

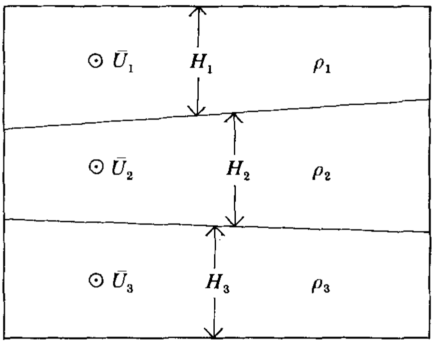

FIGURE 1. Three-layer model. Each layer i ($= 1,2,3$) is of depth H_i, density ρ_i and has mean zonal velocity U_i parallel to the x-axis (out of page).

two-layer stratified flow in a channel of width L_c in which $\bar{U}_i$ was uniform in each layer. Since there is no horizontal shear in this model, the only possible modes of instability are those that release the potential energy stored in the sloping interface. Unstable disturbances will grow if

$$\frac{f^2 L_c^2}{g'(H_1 H_2)^{\frac{1}{2}}} > \tfrac{1}{2}\pi^2, \tag{1}$$

where f is the Coriolis parameter, and $g' = 2g(\rho_2-\rho_1)/(\rho_2+\rho_1)$ is the reduced gravity. When

$$\frac{f^2 L_c^2}{g'(H_1 H_2)^{\frac{1}{2}}} \gg \tfrac{1}{2}\pi^2$$

the wavelength λ_G of the fastest growing mode is

$$\lambda_G \approx 2\pi[g'(H_1 H_2)^{\frac{1}{2}}]^{\frac{1}{2}} f^{-1}, \tag{2}$$

i.e. 2π times the Rossby radius.

Theoretical studies of three-layer flows have been conducted by Davey (1977) and Ikeda (1983). Davey (1977) examined analytically the linear stability of flow in a channel in which the velocity $\bar{U}_i$ was uniform in each layer, $H_1 = H_2 = H_3$ and $\rho_3-\rho_2 = \rho_2-\rho_1$. He found that curvature in the vertical profile of the horizontal velocity could change the range of unstable wavenumbers and the value of the wavenumber of maximum growth rate. These effects were (when the lower boundary was horizontal) parametrized by $S = (\bar{U}_3-\bar{U}_2)/(\bar{U}_2-\bar{U}_1)$.

Ikeda (1983) considered the linear stability of a similar three-layer model, in which $\rho_2-\rho_1 > \rho_3-\rho_2$, $\bar{U}_2 = \bar{U}_3 = 0$ and $H_2 = H_3 = 2H_1$. When the lower boundary was horizontal, the effect of the lower interface upon the range of unstable wavenumbers was small, but when there was a bottom slope the presence of the lower interface allowed unstable modes not present in the equivalent two-layer flow (i.e. that in which $\rho_3 = \rho_2$). Ikeda (1983) also investigated, numerically, the effects of horizontal shear and of a lateral boundary in the three-layer flow.

Somewhat different studies of instability in three-layer flows have been reported by Holmboe (1968) and Wright (1980). Holmboe (1968) discussed the instability of a vertically symmetric, continuously stratified flow. The stratification in each layer was linear but that in the central layer was different from that in the outer layers. Wright (1980) examined a similar problem in which the density had a linear profile in the middle layer, but was uniform in the upper and lower layers.

In the following sections a linear stability analysis similar to those of Davey (1977) and Ikeda (1983), but for variable values of the velocity shears, density differences and layer depths, is discussed. Attention is focused upon the limit of $\epsilon = \Delta\rho_2/\Delta\rho_1 \ll 1$ ($\Delta\rho_1, \Delta\rho_2$ are the density differences across the upper and lower interfaces). The results of this study are used in an accompanying paper to interpret the results of some experiments with three-layer vortices. In the experiments the velocity is not uniform within each layer and the flow does not satisfy the conditions of quasi-geostrophy. However, the successful use by Griffiths & Linden (1981) of the results of Phillips (1954) in the interpretation of similar experiments with two-layer eddies suggested that the present theory could be used to interpret the experiments with three-layer stratification. The applicability of the theory to the experiments is discussed further in Part 2 (Smeed 1988).

2. The model equations

It is assumed that the flow is inviscid and satisfies the quasi-geostrophic equations of motion. The initial state is one of along-channel flow of $\bar{U}_i$ in layer i. The velocity of the disturbance is given by

$$\left(-\frac{\partial\bar{\phi}_i}{\partial\bar{y}}, \frac{\partial\bar{\phi}_i}{\partial\bar{x}}\right).$$

The variables are non-dimensionalized as follows

$$(\bar{x}, \bar{y}) = L(x, y), \quad \bar{U}_i = U_0 U_i,$$

$$\bar{\phi}_i = \mu L U_0 \phi_i, \quad \bar{t} = 2f^{-1}t$$

where L is a horizontal lengthscale and U_0 is a typical value of the along-channel velocity. The reduced gravities at the upper and lower interfaces are g_1' and g_2' where

$$g_1' = 2g\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right), \quad g_2' = 2g\left(\frac{\rho_3-\rho_2}{\rho_3+\rho_2}\right).$$

Assuming that the amplitude μ of the disturbance is small, the equations representing the conservation of potential vorticity in each layer may be linearized to obtain

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_1}[\nabla^2\phi_1 - F_1(\phi_1-\phi_2)] + \frac{\partial\phi_1}{\partial x}\frac{\partial\pi_1}{\partial y} &= 0,\\ \frac{\mathrm{d}}{\mathrm{d}t_2}[\nabla^2\phi_2 - F_{21}(\phi_2-\phi_1) - F_{23}(\phi_2-\phi_3)] + \frac{\partial\phi_2}{\partial x}\frac{\partial\pi_2}{\partial y} &= 0,\\ \frac{\mathrm{d}}{\mathrm{d}t_3}[\nabla^2\phi_3 - F_3(\phi_3-\phi_2)] + \frac{\partial\phi_3}{\partial x}\frac{\partial\pi_3}{\partial y} &= 0,\end{aligned}\right\} \quad (3)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t_i} = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \quad (i = 1, 2, 3),$$

$$\frac{\partial \pi_1}{\partial y} = -F_1 V_1 - \frac{\partial^2 U_1}{\partial y^2},$$

$$\frac{\partial \pi_2}{\partial y} = F_{21} V_1 + F_{23} V_3 - \frac{\partial^2 U_2}{\partial y^2},$$

$$\frac{\partial \pi_3}{\partial y} = -F_3 V_3 - \frac{\partial^2 U_3}{\partial y^2},$$

$$V_1 = U_2 - U_1, \quad V_3 = U_2 - U_3.$$

The parameters

$$F_1 = \frac{f^2 L^2}{g_1' H_1}, \quad F_{21} = \frac{f^2 L^2}{g_1' H_2},$$

$$F_3 = \frac{f^2 L^2}{g_2' H_3}, \quad F_{23} = \frac{f^2 L^2}{g_2' H_2}$$

are all Froude numbers.

The first term in each of (3) represents the rate of change, following the basic flow, of the perturbation potential vorticity. The second represents the cross-channel advection of the potential vorticity of the basic state by the perturbation velocity.

The boundary conditions on ϕ_i are that there can be no flow across the walls of the channel, i.e.

$$\frac{\partial \phi_i}{\partial x} = 0 \quad \text{on } y = 0, \frac{L_c}{L} \quad (i = 1, 2, 3), \tag{4}$$

where L_c is the width of the channel.

An energy equation can be obtained by integrating the product of ϕ_i with (3) over the horizontal plane and it is found that

$$\sum_{i=1}^{3} H_i \frac{\partial}{\partial t} \iint \mathrm{d}x\,\mathrm{d}y\,\tfrac{1}{2}(\nabla \phi_i)^2 + F_1 H_1 \frac{\partial}{\partial t} \iint \mathrm{d}x\,\mathrm{d}y\,\tfrac{1}{2}(\phi_1 - \phi_2)^2 + F_3 H_3 \frac{\partial}{\partial t} \iint \mathrm{d}x\,\mathrm{d}y\,\tfrac{1}{2}(\phi_3 - \phi_2)^2$$
$$= -F_1 H_1 \iint \mathrm{d}x\,\mathrm{d}y\, V_1 \phi_1 \phi_{2x} - F_3 H_3 \iint \mathrm{d}x\,\mathrm{d}y\, V_3 \phi_3 \phi_{2x}$$
$$+ \sum_{i=1}^{3} H_i \iint \mathrm{d}x\,\mathrm{d}y\, U_{iy} \phi_{ix} \phi_{iy}. \tag{5}$$

The terms on the left-hand side of (5) represent the rate of change of the energy of the perturbations. This consists of the kinetic energy in each layer plus the potential energy associated with the upper and lower interfaces. The sources of energy for the growth of the disturbances are represented by the terms on the right-hand side of (5), they are the potential energy stored in the mean slopes of the upper and lower interfaces and the kinetic energy associated with the horizontal shear in each layer.

As in the model of Phillips (1954), it is assumed here that the velocity U_i is uniform in each layer so that the only disturbances that can grow are those that release potential energy from the basic state. Once this assumption has been made, a solution to (3) can be found by letting

$$\phi_i = \hat{\phi}_i \sin ly\, e^{ik(x-ct)}. \tag{6}$$

To satisfy the boundary conditions (4)

$$l = \frac{n\pi L}{L_c} \quad (n = 1, 2, 3, \ldots).$$

Substitution of (6) into (3) yields a set of three linear algebraic equations for $\hat{\phi}_i$:

$$\sum_{i=1}^{3} a_{ji}\hat{\phi}_i = 0 \quad (j = 1, 2, 3). \tag{7}$$

The coefficients a_{ji} are

$$\left.\begin{aligned} a_{11} &= c'\left(K^2+\frac{1}{\delta_1}\right)+K^2, \quad a_{12} = -\frac{1}{\delta_1}(c'+1), \quad a_{13} = 0,\\ a_{21} &= -\frac{c'}{\delta_2}, \quad a_{22} = c'\left[K^2+\frac{1}{\delta_2}\left(1+\frac{1}{\epsilon}\right)\right]+\frac{1}{\delta_2}(1-S), \quad a_{23} = -\frac{c'}{\epsilon\delta_2},\\ a_{31} &= 0, \quad a_{32} = -(c'-S\epsilon)\frac{1}{\epsilon\delta_3}, \quad a_{33} = c'\left(K^2+\frac{1}{\epsilon\delta_3}\right)-S\epsilon K^2, \end{aligned}\right\} \tag{8}$$

where the wave speed and wavenumber have been rescaled,

$$c' = \frac{c-U_2}{V_1}, \quad K^2 = \frac{g_1' H}{f^2 L^2}(k^2+l^2),$$

so that both are $O(1)$ for disturbances associated with the upper interface. The total depth of fluid is H, $S = -\epsilon^{-1}V_3/V_1$ is the ratio the slope of the lower interface to that of the upper interface and

$$\epsilon = \frac{g_2'}{g_1'}, \quad \delta_1 = \frac{H_1}{H}, \quad \delta_3 = \frac{H_3}{H}, \quad \delta_2 = 1-\delta_1-\delta_3.$$

For a given wave vector (k, l) the wave speed is obtained as the eigenvalue of

$$|a_{ij}| = 0.$$

This is a cubic equation for c', i.e.

$$\alpha c'^3+\beta c'^2+\gamma c'+\delta = 0.$$

The coefficients α, β, γ and δ are functions of K^2:

$$\left.\begin{aligned} \alpha &= K^4+K^2\left[\frac{1}{\delta_1}+\frac{1}{\delta_2}+\frac{1}{\epsilon}\left(\frac{1}{\delta_2}+\frac{1}{\delta_3}\right)\right]+\frac{1}{\epsilon}\left[\frac{1}{\delta_1\delta_2}+\frac{1}{\delta_2\delta_3}+\frac{1}{\delta_3\delta_1}\right],\\ \beta &= K^4(1-\epsilon S)+K^2\left\{\frac{1}{\epsilon}\left(\frac{1}{\delta_2}+\frac{1}{\delta_3}\right)+\frac{2}{\delta_2}-\epsilon S\left[\frac{1}{\delta_1}+\frac{1}{\delta_2}+\frac{2}{\epsilon\delta_2}\right]\right\}+\frac{2}{\delta_2\epsilon}\left[\frac{1}{\delta_3}-\frac{\epsilon S}{\delta_1}\right],\\ \gamma &= -K^4\epsilon S+\frac{K^2}{\delta_2}\left[1+\epsilon S^2-2\epsilon S\left(1+\frac{1}{\epsilon}\right)\right]+\frac{1}{\delta_2\epsilon}\left[\frac{1}{\delta_3}+\frac{\epsilon^2 S^2}{\delta_1}\right],\\ \delta &= -\frac{K^2\epsilon S}{\delta_2}(1-S). \end{aligned}\right\} \tag{10}$$

The solutions to (9) may be either three real roots or one real root and a conjugate pair of complex roots. If the latter is true, then the flow will be unstable to small perturbations. The boundary between the stable and unstable modes is given by

$$4\gamma^3\alpha - \gamma^2\beta^2 + 27\delta^2\alpha^2 - 18\alpha\beta\gamma\delta + 4\delta\beta^3 = 0. \tag{11}$$

No simple expression for c' can be obtained from (9) and (10) and so the solutions in the asymptotic limits of $\epsilon \to 0$ and $\delta_3 \to 0$ will be discussed.

Before discussing the particular cases two general remarks can be made. First, if either (or both) of V_1 and V_3 are non-zero there will always be a layer in which the sign of the gradient of potential vorticity will be opposite to that in one (in which case the gradient is zero in the third layer) or both of the other two layers. The necessary condition for instability is thus always satisfied. Secondly, from (9) and (10) it can be shown that for any given values of the parameters, all wavenumbers greater than some critical value are stable.

3. Equal layer depths and equal density differences

The case in which $\delta_1 = \delta_3 = \frac{1}{3}$ and $\epsilon = 1$ has been studied by Davey (1977). In this example, the effect of curvature in the vertical velocity profile may be examined whilst keeping the stratification uniform. The results are illustrated in figure 2.

The value of the largest unstable wavenumber K^* has a minimum as a function of S when $S = 1$, showing that curvature in the velocity profile increases the range of unstable wavenumbers. When $S < 0$ the values of K^* are larger than when $S > 0$. In particular, as $S \uparrow 0$ there is a band of short, unstable waves. Davey (1977) noted that as $K^2 \to \infty$ the two interfaces become decoupled and the interface with the greater slope behaves somewhat like a rigid boundary, thus he was able to explain the behaviour as $S \uparrow 0$ by analogy with two-layer flow over a sloping bottom. In the two-layer problem, the bottom slope causes a large difference between the gradients of potential vorticity in the two layers for even a small vertical shear, and, when the interface slope is opposite to that of the bottom boundary, a small band of unstable wavenumbers is found with $K^2 \to \infty$ as the shear tends to zero.

4. The limit of $\epsilon \ll 1$

In this section the special case in which the reduced gravity of the lower interface is much less than that of the upper interface, i.e. $\epsilon \ll 1$, is considered. By symmetry the results are equally applicable to the case $\epsilon \gg 1$. The layer depths are assumed to be of the same order, though not necessarily equal.

There are two possible scales of instability in this problem. Associated with the upper interface there may be unstable waves with $K = O(1)$ but shorter-scale disturbances (wavenumber $O(\epsilon^{-\frac{1}{2}})$) may grow because of the presence of the lower interface.

Assuming that $S^{-1} = O(1)$, the marginally stable wavenumber K^* may be expanded as

$$K^{*2} = \epsilon_{-1} X_{-1} + X_0 + \dots. \tag{12}$$

The coefficients X_j may be determined by substituting (12) into (10) and (11) and equating terms of the same order in ϵ. Consideration of the terms of lowest order shows that

$$X_{-1} = 0 \quad \text{or} \quad \frac{4X_{-1}^2}{\delta_2} S[S-1]\left(X_{-1} + \frac{1}{\delta_2} + \frac{1}{\delta_3}\right) = \left[SX_{-1}^2 - \frac{(1-2S)X_{-1}}{\delta_2} - \frac{1}{\delta_2\delta_3}\right]^2 \tag{13}$$

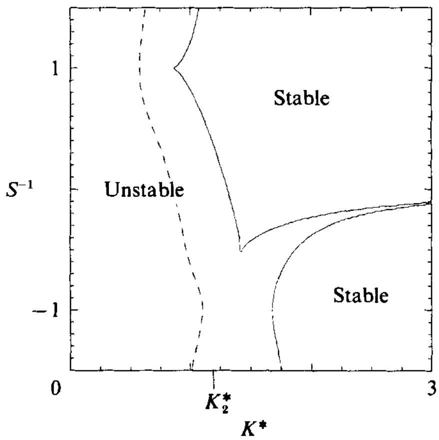

FIGURE 2. Range of unstable wavenumbers as a function of S^{-1}, for the case of equal layer depths and equal reduced gravities ($\epsilon = 1, \delta_1 = \delta_3 = \frac{1}{3}$). The wavenumber of marginal stability when $\epsilon = S = 0, K_2^*$, is indicated on the K^*-axis. The fastest growing mode is indicated (broken line) for the case of an infinitely wide channel (i.e. $l = 0$).

and so it can be seen that $O(\epsilon^{-\frac{1}{2}})$ wavenumber instabilities (i.e. $X_{-1} \neq 0$) are possible only if

$$S^{-1} < 1. \tag{14}$$

This may be explained by noting that when $K^2 \gg 1$ the upper interface acts somewhat like a rigid boundary and that (14) is the requirement that the gradients of potential vorticity in the lower two layers be of opposite sign.

Further examination of (13) reveals that as $S \downarrow 1$ only a finite band of $O(\epsilon^{-\frac{1}{2}})$ wavenumbers are unstable and that as $S \downarrow 1$ the width of this band $\rightarrow 0$.

Thus it may be concluded that as $S \downarrow 1$ there are two bands of unstable wavenumbers, the long waves ($K = O(1)$) and the short waves ($K = O(\epsilon^{-\frac{1}{2}})$). In between these there is a range of wavenumbers stabilized by the upper interface.

It can also be seen that as $S \rightarrow \infty$, (13) implies

$$X_{-1}^2 = \frac{4}{\delta_2 \delta_3}; \tag{15}$$

this is the same value as for the two-layer problem with layer depths H_2 and H_3. In this case the upper interface acts as a rigid boundary.

These features can be seen in figure 3 in which K^* is plotted as a function of S for $\delta_1 = \delta_3 = \frac{1}{3}$, $\epsilon = 0.5$, 0.2, 0.1 and 0.05. Also shown in figure 3 is K_G, the wavenumber of maximum growth rate when $l = 0$ (i.e. in the limit of an infinitely wide channel).

It can also be seen in figure 3 that although unstable waves with $K^2 = O(\epsilon^{-1})$ are possible for all $S > 1$, the growth rates of these disturbances only become equal to those of long waves for $|S|$ significantly greater than 1. For the smaller values of ϵ there is a discontinuity in the value of K_G such that $K_G^2 = O(\epsilon)$ for $S > S_c$, and $K_G^2 = O(1)$ for $S < S_c$. The value of S_c was calculated for the case in which the layers were all of equal depth (figure 4). In this case $S_c \sim 0.8\epsilon^{-\frac{1}{2}}$ as $\epsilon \rightarrow 0$.

To explain some of the features described above, it is helpful to consider the magnitude of the terms in the energy equation (5). First consider P_1 and P_3, the terms

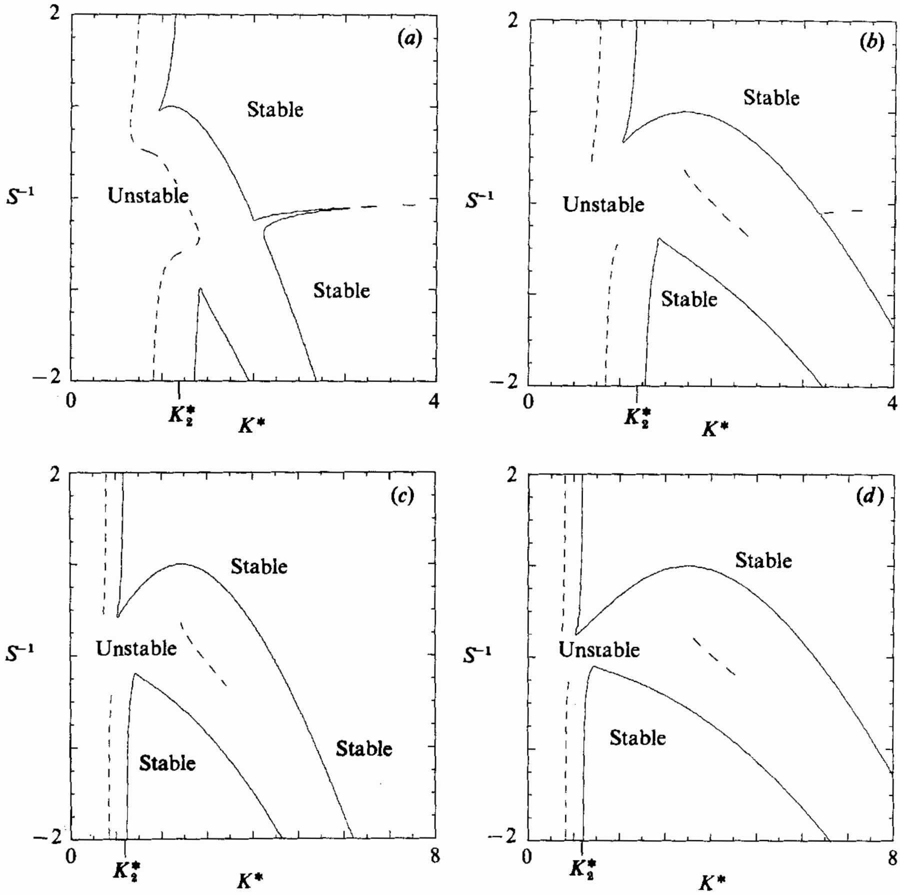

FIGURE 3. Range of unstable wavenumbers as a function of S^{-1}, for the case of equal layer depths but for different reduced gravities. The wavenumber of marginal stability when $\epsilon = S = 0, K_2^*$, is indicated on the K^*-axis. The fastest growing mode, K_G, is indicated (broken line) for the case of an infinitely wide channel (i.e. $l = 0$). $\delta_1 = \delta_3 = \frac{1}{3}$ and (a) $\epsilon = 0.5$, (b) 0.2, (c), 0.1, (d) 0.05.

representing the release of potential energy from the mean slope of the upper and the lower interface respectively:

$$P_j = -H_j F_j V_j \int \mathrm{d}y\, \overline{\phi_j \phi_{2x}} \quad (j = 1, 3) \tag{16}$$

where the overbar represents an average in x and t. Substituting from (6) it is found that

$$\begin{aligned} P_j &= H_j F_j V_j \tfrac{1}{8}\mathrm{i}k[\hat{\phi}_j \hat{\phi}_2^* - \hat{\phi}_j^* \hat{\phi}_2]\, \mathrm{e}^{2kV_1 c_i' t} \\ &= -H_j F_j V_j \tfrac{1}{4}k\, \mathrm{Im}\left[\frac{\hat{\phi}_j}{\hat{\phi}_2}\right] |\hat{\phi}_2|^2\, \mathrm{e}^{2kV_1 c_i' t}, \end{aligned} \tag{17}$$

where c_i' is the imaginary part of c' and $\mathrm{Im}\,(X)$ is the imaginary part of the complex variable X. From (7) and (8)

$$\frac{\hat{\phi}_1}{\hat{\phi}_2} = -\frac{a_{12}}{a_{11}} = \frac{1}{1+\delta_1 K^2}\left\{1 + \frac{1}{[c'(1+\delta_1 K^2) + \delta_1 K^2]}\right\} \tag{18}$$

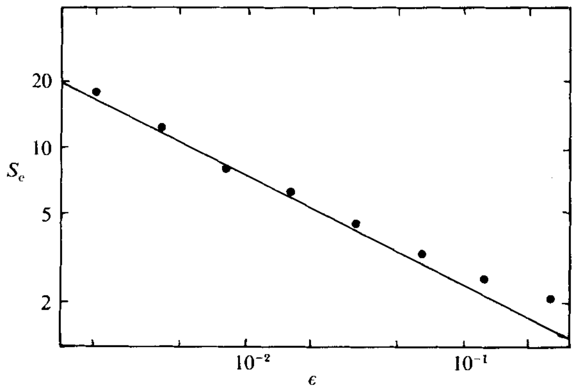

FIGURE 4. The value S_c of S ($S > 0$) at which the wavenumber of maximum growth rate changes from the long waves associated with the upper interface to the short waves associated with the lower interface as a function of ϵ for the case $l = 0, \delta_1 = \delta_3 = \frac{1}{3}$. The straight line is given by $S_c = 0.8\epsilon^{-\frac{1}{2}}$.

and so
$$\operatorname{Im}\left(\frac{\hat{\phi}_1}{\hat{\phi}_2}\right) = -\left(\frac{1}{1+\delta_1 K^2}\right)^2 \frac{c_i'}{c_i'^2+\left[c_r'+\dfrac{\delta_1 K^2}{1+\delta_1 K^2}\right]^2}. \tag{19}$$

Similarly
$$\operatorname{Im}\left(\frac{\hat{\phi}_3}{\hat{\phi}_2}\right) = S\epsilon\left(\frac{1}{1+\epsilon\delta_3 K^2}\right)^2 \frac{c_i'}{c_i'^2+\left[c_r'-\dfrac{S\epsilon^2\delta_3 K^2}{1+\epsilon\delta_3 K^2}\right]^2}. \tag{20}$$

Note that (19) and (20) imply that P_1 and P_3 always have the same sign (they are both positive when $c_i' > 0$). Therefore, it is not possible to have a mode of instability that transfers mean potential energy from one interface to mean potential energy of the other interface for any values of the parameters. The rates of production may though be quite different.

Secondly consider the rates of change of potential energy of the perturbations at the upper interface $\dot{PE}_1$ and at the lower interface $\dot{PE}_3$. These are given by

$$\dot{PE}_j = \tfrac{1}{2}F_j H_j \frac{\partial}{\partial t}\int \mathrm{d}y\overline{(\phi_j-\phi_2)^2} \quad (j = 1,3)$$
$$= \tfrac{1}{4}kV_1 c_i' \, e^{2kV_1c_i't} F_j H_j |\hat{\phi}_2|^2 \left(1-\frac{\hat{\phi}_j}{\hat{\phi}_2}\right)^2. \tag{21}$$

The magnitudes of the terms P_3/P_1, $\dot{PE}_1/P_1$, $\dot{PE}_3/P_3$ and $|(\hat{\phi}_2-\hat{\phi}_3)/(\hat{\phi}_1-\hat{\phi}_2)|$ (the amplitude of the perturbations on the lower interface divided by the amplitude of the perturbations on the upper interface) were estimated in the limit of $\epsilon \to 0$ for $K^2 = O(1)$ and $K^2 = O(\epsilon^{-1})$ (table 1). To calculate these terms it is necessary to know the magnitude of c'. Examination of (9) in the limit of $\epsilon \to 0$ indicates that for $K^2 = O(\epsilon^{-1})$ imaginary values of c' (hence unstable modes) are only possible for $c' = O(\epsilon S)$. When $K^2 = O(1)$ growing disturbances have $c' = O(1)$. In the regime $K^2 = O(1), c' = O(1)$ the amplitude of the disturbance is of the same order on each interface and $P_3/P_1 = S^2\epsilon$ = the ratio of the available potential energy of the lower interface to that of the upper interface. When S is small (large) there is a net transfer from the upper (lower) to the lower (upper) interface. For disturbances with

K^2	c'	$\dfrac{P_3}{P_1}$	$\dfrac{PE_1}{P_1}$	$\dfrac{PE_3}{P_3}$	$\left\lvert\dfrac{\hat{\phi}_3-\hat{\phi}_2}{\hat{\phi}_2-\hat{\phi}_1}\right\rvert$
1	1	$S^2\epsilon$	1	S^{-2}	1
ϵ^{-1}	$S\epsilon$	ϵ^{-3}	ϵ^{-2}	1	ϵ^{-1}

TABLE 1. Orders of magnitudes of the terms in the energy equation (5) in the limit of $\epsilon \to 0$ (S constant). $|(\hat{\phi}_3-\hat{\phi}_2)/(\hat{\phi}_2-\hat{\phi}_1)|$ is the ratio of the amplitude of the perturbations on the lower interface to the amplitude of the perturbations on the upper interface. P_1, P_3, PE_1 and PE_3 are defined in equations (16) and (21).

$K^2 = O(\epsilon^{-1}), c' = O(\epsilon S)$, however, the amplitude of the disturbances on the upper interface is $O(\epsilon)$ times that on the lower interface and most of the energy is released from the mean slope of the lower interface, a proportion $O(\epsilon)$ of which is transferred to perturbations on the upper interface.

The change in the value of the wavenumber of maximum growth from $K_G^2 = O(1)$ when $S < S_c$ to $K_G^2 = O(\epsilon^{-1})$ for $S > S_c$ where $S_c \sim \epsilon^{-\frac{1}{2}}$, may be understood by noting that the ratio of the growth rate kc_i for the long waves with $K^2 = O(1)$ and $c' = O(1)$ to that for the short waves, with $K^2 = O(\epsilon^{-1})$ and $c' = O(S\epsilon)$ is $1/S\epsilon^{\frac{1}{2}}$. Thus the fastest growing waves are those associated with the interface with the greatest amount of available potential energy.

5. A thin lower layer

Another case of interest in which there are two possible scales of instability, is that in which the bottom layer is much shallower than the upper two (or equivalently the lower two layers are much deeper than the top layer). A stratification such as this could be used to model the bottom mixed layer of the benthic boundary layer below a deep stratified layer or the ocean thermocline above a deep stratified layer.

It is assumed that g_1' and g_2' are of the same order, though not necessarily equal ($\epsilon \sim 1$), and that $\delta_3 \ll \delta_1, \delta_2$. In this case short waves associated with the lower interface may be expected to have $K^2 = O(\delta^{-\frac{1}{2}})$, and long waves associated with the top two layers will have $K^2 = O(1)$. The range of unstable wavenumbers may be examined in a similar manner to that used in §4, i.e. by expanding K^{*2} as

$$K^{*2} = \delta_3^{-\frac{1}{2}} X_{-1} + X_0 + \ldots, \tag{22}$$

substituting (22) into (10) and (11), and equating terms of the same order in δ_3.

Inspection of the terms of lowest order in δ_3 show that X_{-1} is given by

$$X_{-1} = 0 \quad \text{or} \quad \frac{4X_{-1}^2}{\delta_2} S[S-1] = \left(X_{-1}^2 \epsilon S - \frac{1}{\epsilon\delta_2}\right)^2. \tag{23}$$

It is thus apparent that short waves are only possible for $S^{-1} < 1$. Equation (23) also indicates that as $S \downarrow 1$, there is only a finite band of unstable wavenumbers with $K^2 = O(\delta_3^{-\frac{1}{2}})$ and the width of this band $\to 0$ as $S \downarrow 1$. It may be concluded that as $S \downarrow 1$ there are two bands of unstable wavenumbers, the long waves with $K^2 = O(1)$ and the shorter waves with $K^2 = O(\delta_3^{-\frac{1}{2}})$. In between these there is a band of stable wavenumbers.

These features are illustrated in figure 5, in which K^* is plotted as a function of S for $\epsilon = 1$, $\delta_1 = \delta_2$, $\delta_3/\delta_1 = 0.5$, 0.2, 0.1 and 0.05. There is a discontinuity in the

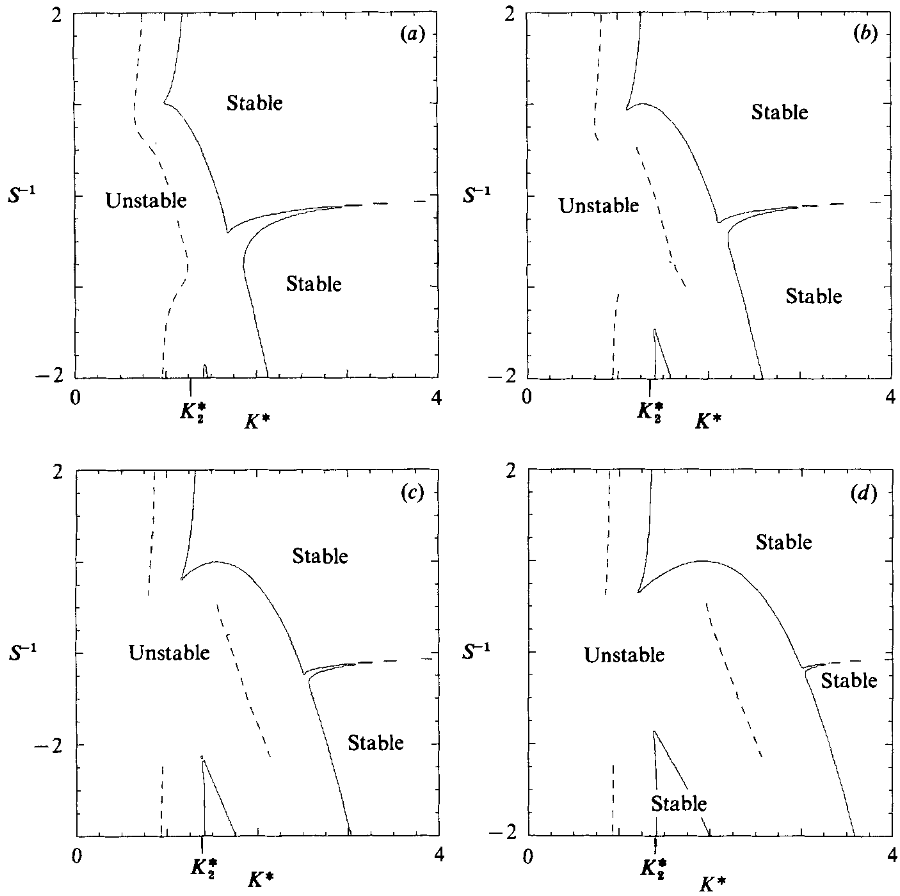

FIGURE 5. Range of unstable wavenumbers as a function of S^{-1}, for the case $\epsilon = 1, \delta_1 = \delta_2$ and (*a*) $\delta_3/\delta_1 = 0.5$, (*b*) 0.2, (*c*) 0.1, (*d*) 0.05. The wavenumber of marginal stability when $\delta_3 = 0, K_2^*$, is indicated on the K^*-axis. The fastest growing mode, K_G, is indicated (broken line) for the case of an infinitely wide channel (i.e. $l = 0$).

wavenumber of maximum growth rate, however, the critical value S_c of S appears to decrease only slowly with δ_3 for the range of parameters in figure 5. Examination of (9) in the limit of $\delta_3 \rightarrow 0$ indicates that growing disturbances with $K^2 = O(\delta_3^{-\frac{1}{2}})$ have $c' = O(S\delta_3^{\frac{1}{4}})$. Thus the growth rates are $O(S\delta_3^{\frac{1}{4}})$ indicating that $S_c \sim \delta_3^{-\frac{1}{4}}$, which does in fact appear to be the case for $\delta_3 < 0.05$ (figure 6).

6. The stability of flows with continuous density profiles

This study was intended primarily to model flows characterized by a strong pycnocline above a weakly stratified layer, the second interface being a crude representation of the stratification below the pycnocline. Warm ocean eddies such as Gulf Stream rings often have density profiles of this form and an example is discussed in Smeed (1988). The results are, though, also relevant to flows with more general density profiles.

Flows with continuous density profiles (described by $(1/\rho g)(\partial\rho/\partial z) = N^2(z)$)

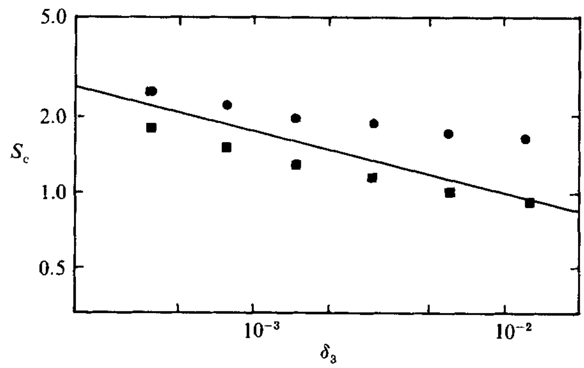

FIGURE 6. The value S_c of S at which the wavenumber of maximum growth rate changes from the long waves associated with the upper interface to the short waves associated with the lower interface as a function of δ_3 for the case $l = 0, \delta_1 = \delta_2, \epsilon = 1$. The straight line is given by $S_c = 10^{\frac{1}{2}}\cdot\delta^{-\frac{1}{4}}$. ● is the transition for $S > 0$ and ■ the absolute value of the transition for $S < 0$.

satisfying the quasi-geostrophic equations of motion may be considered as the sum of a set of orthogonal normal modes, so that the pressure is given by

$$p = \sum_j A_j(x, y, t)\, Q_j(z), \tag{24}$$

where Q_j satisfy

$$\left.\begin{aligned} &\frac{\partial}{\partial z} Q_j = 0 \quad \text{on } z = 0, 1, \\ &\frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial}{\partial z} Q_j + \Lambda_j Q_j = 0, \\ &\int_0^1 \mathrm{d}z\, Q_i Q_j = \delta_{ij} \end{aligned}\right\} \tag{25}$$

(see e.g. Flierl 1978). If this system is truncated to consider only the first N modes, a set of N equations is obtained for A_j $(j = 0, 1, \dots, N-1)$.

$$\frac{\partial}{\partial t}(\nabla_{\mathrm{H}}^2 - \Lambda_j) A_j + \sum_{i,k} \Gamma_{ijk}\, \mathrm{J}[A_i, (\nabla_{\mathrm{H}}^2 - \Lambda_k) A_k] = 0, \tag{26}$$

where ∇_{H}^2 is the horizontal Laplacian, J is the Jacobean operator, and

$$\Gamma_{ijk} = \int_0^1 \mathrm{d}z\, Q_i Q_j Q_k. \tag{27}$$

Normal modes may also be defined for three-layer quasi-geostrophic flow. These modes satisfy

$$\left.\begin{aligned} -\frac{1}{\delta_1}(q_j^1 - q_j^2) + \lambda_j q_j^1 &= 0, \\ \frac{1}{\delta_2}(q_j^1 - q_j^2) - \frac{1}{\epsilon\delta_2}(q_j^2 - q_j^3) + \lambda_j q_j^2 &= 0, \\ \frac{1}{\epsilon\delta_3}(q_j^2 - q_j^3) + \lambda_j q_j^3 &= 0, \\ \sum_{i=1}^{3} \delta_i q_j^i q_k^i &= \delta_{jk}, \end{aligned}\right\} \tag{28}$$

where q_j^i is the normalized amplitude of the jth mode in layer i, so that the pressure in layer i is given by

$$p_i = \sum_{j=0}^{2} a_j(x,y,t)\, q_j^i. \tag{29}$$

Note that λ_i is non-dimensionalized by $f^2L^2/g_1' H$. The functions a_j satisfy the same set of equations as the truncated set of modes A_j but with Γ_{ijk} replaced by γ_{ijk} and Λ_j replaced by λ_j, where

$$\gamma_{ijk} = \sum_{n=1}^{3} \delta_n q_i^n q_j^n q_k^n. \tag{30}$$

If λ_j and γ_{ijk} take the same values as Λ_j and Γ_{ijk} then the layer flow models exactly the truncated set of modes of the continuously stratified flow. In the problem examined here, in which the mean flow varies only with z, the barotropic mode may be neglected. It is thus required to set the values of λ_2/λ_1, γ_{111}, γ_{112}, γ_{122} and γ_{222}. However, there are only three variables, ϵ, δ_1 and δ_3, that can be varied in the three-layer model. Thus, in general, it is not possible to represent exactly the truncated system (26). This problem has been highlighted by Flierl (1978) who discussed in detail the calibration of two-layer models. The qualitative conclusions drawn from this study of three-layer flow do, though, give some insight into continuously stratified flows.

The eigenvalues λ_j (which are the inverse squares of the deformation radii) satisfy)

$$\left.\begin{aligned} &\lambda_0 = 0, \quad \lambda_1 = \tfrac{1}{2}[-b-(b^2-4c)^{\frac{1}{2}}], \quad \lambda_2 = \tfrac{1}{2}[-b+(b^2-4c)^{\frac{1}{2}}], \\ &b = -\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}+\frac{1}{\epsilon\delta_2}+\frac{1}{\epsilon\delta_3}\right), \quad c = \frac{1}{\epsilon\delta_1\delta_2\delta_3}. \end{aligned}\right\} \tag{31}$$

When $\epsilon \ll 1$ these are given approximately by

$$\lambda_1 = \frac{1}{\delta_1(1-\delta_1)}, \quad \lambda_2 = \frac{1-\delta_1}{\epsilon\delta_2\delta_3}. \tag{32}$$

The functions a_j corresponding to the uniform mean flow considered here may be expressed as $a_j = a_{j0} + \hat{a}_j y$ and so the ratio of the slope of the lower interface to that of the upper interface is given by

$$S = \frac{1}{\epsilon}\left\{\frac{\hat{a}_1(q_1^2-q_1^3)+\hat{a}_2(q_2^2-q_2^3)}{\hat{a}_1(q_1^1-q_1^2)+\hat{a}_2(q_2^1-q_2^2)}\right\}. \tag{33}$$

Expression (33) for S may be evaluated using (28). When $\epsilon \ll 1$ it can be shown that

$$\left.\begin{aligned} &\hat{a}_2 = 0 \Rightarrow S = \frac{\delta_3}{1-\delta_1} \Rightarrow 0 < S < 1, \\ &\hat{a}_1 = 0 \Rightarrow S = -\frac{(1-\delta_1)}{\epsilon\delta_3} \Rightarrow S < -1, \end{aligned}\right\} \tag{34}$$

and if $\hat{a}_1$ and $\hat{a}_2$ are both non-zero

$$S = \frac{\delta_3}{1-\delta_1}\left\{\frac{1+\dfrac{\hat{a}_2}{\hat{a}_1}O(\epsilon^{-1})}{1+\dfrac{\hat{a}_2}{\hat{a}_1}O(1)}\right\}. \tag{35}$$

In the three-layer model wavenumbers $O(\epsilon^{-\frac{1}{2}})$ were unstable only when $|S| > 1$, and these modes had the largest growth rates when $|S| \gtrsim \epsilon^{-\frac{1}{2}}$. Equation (35) would then indicate that in a continuously stratified flow in which the deformation radius of the second baroclinic mode is $\epsilon^{\frac{1}{2}}$ times the deformation radius of the first baroclinic mode, short-scale disturbances are unstable if $|\hat{a}_2/\hat{a}_1| \gtrsim \epsilon$. For the short waves to be the fastest growing modes it is necessary for $|\hat{a}_2/\hat{a}_1| \gtrsim 1$.

7. Summary

The linear stability of quasi-geostrophic three-layer stratified flows in which the horizontal velocity is uniform in each layer has been examined. In this model the only source of energy for growing disturbances is the potential energy stored in the sloping interfaces. Attention has been focused upon the limit of $\epsilon = \Delta\rho_2/\Delta\rho_1 \ll 1$. In this limit two scales of unstable motions are possible: long waves with $K^2 = O(1)$ associated with the upper interface, and short waves ($K^2 = O(\epsilon^{-1})$) associated with the lower interface. The stability of the flow was examined as a function of S, the ratio of the slope of the lower interface to the slope of the upper interface. The results in the limit of $\epsilon \rightarrow 0$ may be summarized as follows.

$0 < S < 1$

Only long-wave disturbances associated with the upper interface are unstable. The amplitude of perturbations is of the same order on the two interfaces.

$1 < |S| < \epsilon^{-\frac{1}{2}}$

Short waves associated with the lower interface are also unstable but the growth rates are small compared with the long waves. There is also an intermediate range of stable wavenumbers. The amplitude of short-wave disturbances is small ($O(\epsilon)$) on the upper interface.

$|S| > \epsilon^{-\frac{1}{2}}$

The short waves have the largest growth rates.

$S \rightarrow -\infty$

There is a band of short unstable wavenumbers $K^2 \rightarrow \infty$. However, the growth rates are small. The width of this band decreases as $\epsilon \rightarrow 0$.

$S \uparrow 0$

There is a band of short unstable wavenumbers $K^2 \rightarrow \infty$. However, the growth rates are small.

Examination of terms in the energy equation shows that all growing disturbances release energy from both interfaces. The rates of production, though, may be quite different and there may be a net transfer in the total potential energy.

The limit of a thin lower layer below two deep layers has also been examined. Short-wave instabilities ($K^2 = O(\delta^{-\frac{1}{2}})$) associated with the lower layer were found for $|S| > 1$.

The results of this analysis are used in an accompanying paper (Smeed 1988) to interpret the results of some experiments with three-layer eddies.

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